ON THE STABLE HOMOLOGY OF MODULES

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Abstract. In this paper, we study certain properties of the stable homology groups of modules over an associative ring, which were defined by Vogel [12]. We compute the kernel of the natural surjection from stable homology to complete homology, which was itself defined by Triulzi [21]. This computation may be used in order to formulate conditions under which the two theories are isomorphic. Duality considerations reveal a connection between stable homology and the complete cohomology theory defined by Nucinkis [19]. Using this connection, we show that the vanishing of the stable homology functors detects modules of finite flat or injective dimension over Noetherian rings. As another application, we characterize the coherent rings over which stable homology is balanced, in terms of the finiteness of the flat dimension of injective modules.

Contents

0. Introduction

The classical Tate cohomology theory of finite groups (cf. [3, Chapter XII]) has been generalized to modules over any ring R by Mislin [18]. Using an approach that involves the notion of satellites, he defined for any left R-module M the complete cohomology functor $\widehat{\operatorname{Ext}}^*_R(M,\_)$ and a natural transformation $\operatorname{Ext}^*_R(M,\_) \longrightarrow \widehat{\operatorname{Ext}}^*_R(M,\_)$ as the projective completion of the ordinary Ext functor $\operatorname{Ext}^*_R(M,\_)$; Mislin's approach is heavily influenced by Gedrich and Gruenberg's theory of terminal completions [11]. As an immediate consequence of the definition, we note that the complete cohomology groups $\widehat{\operatorname{Ext}}^*_R(M,N)$ vanish if $\operatorname{pd}_R N < \infty$. Equivalent descriptions of the complete cohomology functors have been independently formulated by Vogel in [12] and by Benson and Carlson in [2]. Using the approach by Benson and Carlson, it follows that the elements in the kernel of the canonical map $\operatorname{Hom}_R(M,N) \longrightarrow \widehat{\operatorname{Ext}}^0_R(M,N)$ are those R-linear maps $f : M \longrightarrow N$, which are such that the map $\Omega^n f : \Omega^n M \longrightarrow \Omega^n N$ induced by f between the n-th syzygy modules of M and N factors through a projective

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module for $n \gg 0$. In particular, M has finite projective dimension if and only if the complete cohomology functor $\widehat{\mathrm{Ext}}^*_R(M, \_)$ is identically zero; in fact, it only suffices to assume that the group $\widehat{\mathrm{Ext}}^0_R(M, M)$ be trivial (cf. [16, §4.2]).

Using the dual approach, Nucinkis has defined in [19] for any left R-module N the complete cohomology functor $\widetilde{\mathrm{Ext}}^*_R(\_, N)$ and a natural transformation $\mathrm{Ext}^*_R(\_, N) \longrightarrow \widetilde{\mathrm{Ext}}^*_R(\_, N)$, as the injective completion of the Ext functor $\mathrm{Ext}^*_R(\_, N)$. The complete cohomology groups $\widetilde{\mathrm{Ext}}^*_R(M, N)$ vanish if M has finite injective dimension. Equivalent descriptions of these functors may be obtained by using the dual approaches to those by Vogel and Benson and Carlson. It follows that the elements in the kernel of the canonical map $\mathrm{Hom}_R(M, N) \longrightarrow \widetilde{\mathrm{Ext}}^0_R(M, N)$ are those R-linear maps $f : M \longrightarrow N$, which are such that the map $\Sigma^n f : \Sigma^n M \longrightarrow \Sigma^n N$ induced by f between the n-th cosyzygy modules of M and N factors through an injective module for $n \gg 0$. In particular, N has finite injective dimension if and only if the complete cohomology functor $\widetilde{\mathrm{Ext}}^*_R(\_, N)$ is identically zero; in fact, it only suffices to assume that the group $\widetilde{\mathrm{Ext}}^0_R(N, N)$ be trivial (cf. [19, Theorem 3.7]).

If M, N are two left R-modules, then a natural question to ask is whether the complete cohomology groups $\widehat{\mathrm{Ext}}^*_R(M, N)$ and $\widetilde{\mathrm{Ext}}^*_R(M, N)$ are isomorphic to each other. The inherent asymmetry in the definition of these groups suggests that a positive answer to this question shouldn't come for free; instead, it should reflect the presence of a homological finiteness condition of some kind. This is indeed the case: Nucinkis has proved in [19, Theorem 5.2] that the following conditions are equivalent on R:

(i) the groups $\widehat{\mathrm{Ext}}^*_R(M, N)$ and $\widetilde{\mathrm{Ext}}^*_R(M, N)$ are (naturally) isomorphic for all left R-modules M, N and

(ii) any projective (resp. injective) left R-module has finite injective (resp. projective) dimension.

If condition (ii) holds, then (as shown by Gedrich and Gruenberg in [11]) the invariants $\mathrm{silp}\, R$, the supremum of the injective lengths of projective left R-modules, and $\mathrm{spli}\, R$, the supremum of the projective lengths of injective left R-modules, are both finite and equal to each other. The common value of these invariants is closely related to the finiteness of the so-called Gorenstein projective and injective dimensions of left R-modules; the reader may consult [8, §4] for more details on this issue.

Things appear to be more complicated for homology. Given a right R-module M and a left R-module N, Vogel has defined in [12] the stable homology groups $\widetilde{\mathrm{Tor}}^R_*(M, N)$, by using a projective (or flat) resolution of M and an injective resolution of N. It is an immediate consequence of their definition, that the groups $\widetilde{\mathrm{Tor}}^R_*(M, N)$ vanish if M has finite flat dimension or else if N has finite injective dimension. Motivated by the corresponding results concerning complete cohomology, one may ask whether these homological finiteness conditions are equivalent to the vanishing of the functors $\widetilde{\mathrm{Tor}}^R_*(M, \_)$ and $\widetilde{\mathrm{Tor}}^R_*(\_, N)$. Some special cases of this problem have been examined in [4]. We shall prove the following result, which gives an affirmative answer to both questions, in the case of Noetherian rings.

Theorem A. *Let R be a left Noetherian ring.*

(i) A right R-module M has finite flat dimension if and only if the functors $\widetilde{Tor}^R_*(M, \_)$ are identically zero.

(ii) A left R-module N has finite injective dimension if and only if the functors $\widetilde{Tor}_*^R(\_,N)$ are identically zero.

Given a right R-module M and a left R-module N, we may define the stable homology groups $\widetilde{\mathrm{Tor}}_*^{R^o}(N,M)$, using an injective resolution of M and a projective (or flat) resolution of N. A natural question to ask is whether the stable homology groups $\widetilde{\mathrm{Tor}}_*^R(M,N)$ and $\widetilde{\mathrm{Tor}}_*^{R^o}(N,M)$ are isomorphic to each other, i.e. whether stable homology is balanced. As in the case of complete cohomology, the inherent asymmetry in the definition of these groups suggests that a homological finiteness condition of some kind should be involved in the answer. In this direction, we shall prove the following result.

Theorem B. *Let R be a ring, which is left and right coherent. Then, the following conditions are equivalent:*

(i) The stable homology groups $\widetilde{Tor}_*^R(M,N)$ and $\widetilde{Tor}_*^{R^o}(N,M)$ are (naturally) isomorphic for any right R-module M and any left R-module N.

(ii) Any injective left or right R-module has finite flat dimension.

The implication (ii)$\rightarrow$(i) is valid over any ring; this is essentially shown in [4, Theorem 4.2]. We note that if condition (ii) holds, then (as shown in [9, Corollary 2.5] and under no assumption on R) the invariants $\mathrm{sfli}\,R$ and $\mathrm{sfli}\,R^o$, the suprema of the flat lengths of injective left (resp. right) R-modules are both finite and equal to each other. The common value of these invariants is closely related to the finiteness of the so-called Gorenstein flat dimensions of left and right R-modules; the reader may consult [8, §5] for more details on this issue.

Our main technical tool for proving Theorems A and B is a description of the stable homology groups, in terms of a certain inverse system of abelian groups. In order to be more precise, we note that Triulzi has defined in [21] for any right R-module M the complete homology functor $\widehat{\mathrm{Tor}}_*^R(M,\_)$ and a natural transformation $\widehat{\mathrm{Tor}}_*^R(M,\_) \longrightarrow \mathrm{Tor}_*^R(M,\_)$, as the injective completion of the Tor functor $\mathrm{Tor}_*^R(M,\_)$. The complete homology groups $\widehat{\mathrm{Tor}}_*^R(M,N)$ vanish if the left R-module N has finite injective dimension. It turns out that the group $\widehat{\mathrm{Tor}}_n^R(M,N)$ may be computed as the limit of the inverse system $(\mathrm{Tor}_{n+i}^R(M,\Sigma^i N))_i$, where $\Sigma^i N$ denotes the i-th cozyzygy module of N. The universal property of complete homology provides us with a natural map $\widetilde{\mathrm{Tor}}_*^R(M,N) \longrightarrow \widehat{\mathrm{Tor}}_*^R(M,N)$, which is always surjective; this has been proved by Triulzi. We shall identify its kernel as a certain $\varprojlim^1$-group; it will turn out that the kernel in degree n is the group $\varprojlim_i{}^1 \mathrm{Tor}_{n+i+1}^R(M,\Sigma^i N)$. First of all, this computation gives a systematic way of examining conditions under which stable homology is isomorphic with complete homology. In this way, we are able to reinterpret some of the results obtained in [5, §2]. On the other hand, this description of the stable homology groups may be also used in order to analyze their vanishing. This circle of ideas, coupled with the duality between stable homology and Nucinkis' complete cohomology, will lead to the proofs of Theorems A and B.

The contents of the paper are as follows: In Section 1, we record a description of Nucinkis' complete cohomology, in terms of a decreasing filtration of the total Hom complex, which is associated with injective resolutions of both variables. This description will be used in order to establish the duality with stable homology. In the next section, we examine the kernel of the natural map from stable to complete homology, by using a description of the former, in terms of a decreasing filtration of the total tensor product complex, which is associated with a flat (resp. injective) resolution of the first (resp. second) variable. We use some generalities involving the homology of complexes endowed with such decreasing filtrations, that have been collected in the Appendix at the end of the paper. In Section 3, we examine conditions that imply the vanishing of the $\varprojlim^1$-term appearing as the kernel of the natural map from stable to complete homology. In Sections 4 and 5, we examine the vanishing of the stable homology groups and prove Theorem A, by using the duality between stable homology and Nucinkis' complete cohomology. Finally, in Section 6, we study the balance of stable homology and use the vanishing criteria obtained in the previous sections, in order to prove the characterization presented in Theorem B.

Notations and terminology.

(i) All direct and inverse systems examined in this paper are indexed by the ordered set $\mathbb{N}$ of natural numbers.

(ii) If R is a ring, then we denote by R^o its opposite ring. We do not distinguish between left (resp. right) R-modules and right (resp. left) R^o-modules.

(iii) For any left R-module M we shall denote by ΩM the kernel of an epimorphism from a projective module onto M. Even though ΩM is not uniquely determined by M, Schanuel's lemma implies uniqueness up to projective direct summands. The iterates $\Omega^i M$, $i \geq 0$, are defined inductively by letting $\Omega^0 M = M$ and $\Omega^i M = \Omega\Omega^{i-1} M$ for all $i > 0$. If $P_* \longrightarrow M \longrightarrow 0$ is a projective resolution of M, then $\Omega^i M$ is identified with the image of the differential $P_i \longrightarrow P_{i-1}$ for all $i > 0$.

(iv) Dually, for any left R-module M we shall denote by ΣM the cokernel of a monomorphism from M into an injective module; Schanuel's lemma implies uniqueness up to injective direct summands. The iterates $\Sigma^i M$, $i \geq 0$, are defined inductively by letting $\Sigma^0 M = M$ and $\Sigma^i M = \Sigma\Sigma^{i-1} M$ for all $i > 0$. If $0 \longrightarrow M \longrightarrow I^*$ is an injective resolution of M, then $\Sigma^i M$ is identified with the image of the differential $I^{i-1} \longrightarrow I^i$ for all $i > 0$.

(v) We denote by D the Pontryagin duality functors from the category of left (resp. right) R-modules to the category of right (resp. left) R-modules, which are both defined by letting $M \mapsto \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$. We shall often use in the sequel Lambek's equality $\mathrm{fd}_{R^o} M = \mathrm{id}_R DM$, which is valid for any right R-module M; cf. [17, Theorem 4.9]. We shall also use the fact that the functor D maps injective right R-modules onto flat left R-modules, in the special case where the ring R is right coherent; for a proof of that assertion, the reader may consult [22, Lemma 3.1.4].

(vi) For any complex X of left R-modules and any integer i, we shall denote by $X[-i]$ the i-fold suspension of X. In particular, if M is a left R-module, then we denote by $M[i]$ the complex consisting of M in degree i and zeroes elsewhere.

1. The injective completion of the contravariant Hom functor

Let R be a ring and consider two left R-modules N, L. Nucinkis has defined in [19] the complete cohomology groups $\widetilde{\mathrm{Ext}}^*_R(N, L)$, by evaluating the injective completion $\widetilde{\mathrm{Ext}}^*_R(\_, L)$ of the cohomological functor $\mathrm{Ext}^*_R(\_, L)$ to N. These groups vanish if N is injective and are universal (in a certain sense) with that property.

Nucinkis' construction of these complete cohomology groups involves the left satellites of the contravariant Ext functor $\mathrm{Ext}^*_R(\_, L)$, a notion due to Cartan and Eilenberg [3, Chapter III]. There is a concrete description of complete cohomology, which may be expressed in terms of an injective resolution $0 \longrightarrow N \longrightarrow I^*$ of N. If $\Sigma^i N$ is the corresponding i-th cosyzygy

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module of N for all $i \geq 0$, then the short exact sequence

$$0 \longrightarrow \Sigma^i N \longrightarrow I^i \longrightarrow \Sigma^{i+1} N \longrightarrow 0$$

induces connecting homomorphisms

$$d_n^i : \mathrm{Ext}_R^{n+i}(\Sigma^i N, L) \longrightarrow \mathrm{Ext}_R^{n+i+1}(\Sigma^{i+1} N, L) \tag{1}$$

for all n, i with $i \geq 0$. We may consider the direct system $(\mathrm{Ext}_R^{n+i}(\Sigma^i N, L))_i$ with structure maps given by the connecting homomorphisms d_n^i as above; it turns out that

$$\widetilde{\mathrm{Ext}}_R^n(N, L) = \varinjlim_i \mathrm{Ext}_R^{n+i}(\Sigma^i N, L) \tag{2}$$

for all n (see the proof of [19, Theorem 3.6]). The complete cohomology groups defined above admit two other equivalent descriptions: one of them follows an idea by Benson and Carlson [2] and the other one is due to Vogel [12]. Using the approach by Benson and Carlson, one may prove that $\widetilde{\mathrm{Ext}}_R^0(N, N) = 0$ if and only if $\mathrm{id}_R N < \infty$ (cf. [19, Theorem 3.7]).

In the sequel, we shall use yet another description of Nucinkis' complete cohomology groups $\widetilde{\mathrm{Ext}}_R^*(N, L)$, which may be formulated in terms of an injective resolution $0 \longrightarrow N \longrightarrow I^*$ of N and an injective resolution $0 \longrightarrow L \longrightarrow J^*$ of L. For all $i \geq 0$ we consider the subcomplex $I^{\geq i} \subseteq I^*$, which coincides with I in (cohomological) degrees $j \geq i$ and vanishes in degrees $j < i$. We note that the exactness of the sequence

$$0 \longrightarrow \Sigma^i N \longrightarrow I^i \longrightarrow I^{i+1} \longrightarrow \dots$$

may be reformulated by saying that the inclusion $\Sigma^i N \hookrightarrow I^i$ induces a quasi-isomorphism $\Sigma^i N[i] \longrightarrow I^{\geq i}$ for all $i \geq 0$. The short exact sequence of cochain complexes of left R-modules

$$0 \longrightarrow I^{\geq i+1} \longrightarrow I^{\geq i} \longrightarrow I^i[i] \longrightarrow 0$$

induces a short exact sequence of complexes of abelian groups

$$0 \longrightarrow \mathrm{Hom}_R\big(I^i[i], J^*\big) \longrightarrow \mathrm{Hom}_R\big(I^{\geq i}, J^*\big) \xrightarrow{c_i} \mathrm{Hom}_R\big(I^{\geq i+1}, J^*\big) \longrightarrow 0.$$

We note that the cochain map denoted by c_i above is just the restriction along the inclusion $I^{\geq i+1} \subseteq I^{\geq i}$. We are interested in the additive maps

$$c_i^n : H^n\big(\mathrm{Hom}_R\big(I^{\geq i}, J^*\big)\big) \longrightarrow H^n\big(\mathrm{Hom}_R\big(I^{\geq i+1}, J^*\big)\big),$$

which are induced in cohomology by the cochain map c_i for all n, i with $i \geq 0$. These induce a direct system of abelian groups $\big(H^n\big(\mathrm{Hom}_R\big(I^{\geq i}, J^*\big)\big)\big)_i$, whose colimit we describe below.

Proposition 1.1. *Let the notation be as above.*

(i) The quasi-isomorphism $\Sigma^i N[i] \longrightarrow I^{\geq i}$ induces an isomorphism of abelian groups

$$H^n\big(Hom_R\big(I^{\geq i}, J^*\big)\big) \xrightarrow{\sim} Ext_R^{n+i}(\Sigma^i N, L)$$

for all n, i with $i \geq 0$.

(ii) There is a commutative diagram of abelian groups

$$\begin{array}{ccc} H^n\big(Hom_R\big(I^{\geq i}, J^*\big)\big) & \xrightarrow{\sim} & Ext_R^{n+i}(\Sigma^i N, L) \\ c_i^n \downarrow & & \downarrow d_i^n \\ H^n\big(Hom_R\big(I^{\geq i+1}, J^*\big)\big) & \xrightarrow{\sim} & Ext_R^{n+i+1}(\Sigma^{i+1} N, L) \end{array}$$

for all n, i with $i \geq 0$. Here, the horizontal isomorphisms are those defined in (i), whereas d_i^n is the connecting homomorphism (1).

(iii) There is a natural isomorphism $\widetilde{Ext}_R^n(N, L) \simeq \varinjlim_i H^n\big(Hom_R\big(I^{\geq i}, J^*\big)\big)$ for all n.

For the proof, we shall need the following auxiliary result.

Lemma 1.2. *Let J^* be a left bounded cochain complex of injective left R-modules.*

(i) For any acyclic complex of left R-modules C^* the complex of abelian groups $Hom_R(C^*, J^*)$ is acyclic as well.

(ii) Any quasi-isomorphism $f : X^* \longrightarrow Y^*$ of complexes of left R-modules induces a quasi-isomorphism $f^t : Hom_R(Y^*, J^*) \longrightarrow Hom_R(X^*, J^*)$ of complexes of abelian groups.

Proof. (i) A cocycle of degree n of the complex $\mathrm{Hom}_R(C^*, J^*)$ is precisely a cochain map form C^* to the n-fold suspension of J^*. Moreover, the cocycles which are coboundaries are precisely those cochain maps that are null-homotopic. It follows that the acyclicity of $\mathrm{Hom}_R(C^*, J^*)$ is merely a reformulation of the fact that any cochain map from an acyclic complex to a left bounded complex of injective modules is null-homotopic.

(ii) This is an immediate consequence of (i), since the mapping cone $C^* = \mathrm{cone}(f)$ of the quasi-isomorphism f is acyclic and the complex $\mathrm{Hom}_R(C^*, J^*)$ is identified with the mapping cone of the cochain map $f^t : \mathrm{Hom}_R(Y^*, J^*) \longrightarrow \mathrm{Hom}_R(X^*, J^*)$. □

Proof of Proposition 1.1. (i) Since J^* is a left bounded cochain complex of injective modules, Lemma 1.2(ii) implies that the quasi-isomorphism $\Sigma^i N[i] \longrightarrow I^{\geq i}$ induces a quasi-isomorphism

$$\mathrm{Hom}_R\big(I^{\geq i}, J^*\big) \longrightarrow \mathrm{Hom}_R\big(\Sigma^i N[i], J^*\big)$$

for all $i \geq 0$. This finishes the proof, since

$$H^n\big(\mathrm{Hom}_R\big(\Sigma^i N[i], J^*\big)\big) = H^{n+i}\big(\mathrm{Hom}_R\big(\Sigma^i N, J^*\big)\big) = \mathrm{Ext}_R^{n+i}(\Sigma^i N, L)$$

for all n, i with $i \geq 0$.

(ii) The isomorphism $H^n\big(\mathrm{Hom}_R\big(I^{\geq i}, J^*\big)\big) \simeq \mathrm{Ext}_R^{n+i}(\Sigma^i N, L)$ established in (i) identifies an element $\xi \in \mathrm{Ext}_R^{n+i}(\Sigma^i N, L)$, which is represented by a linear map $f : \Sigma^i N \longrightarrow \Sigma^{n+i} L$, with the class of the (unique up to homotopy) cochain map $(f^j)_{j \geq i} : I^{\geq i} \longrightarrow J^*[-n]$ lifting f

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Sigma^i N & \longrightarrow & I^i & \longrightarrow & I^{i+1} & \longrightarrow & I^{i+2} & \longrightarrow & \dots \\
 & & {\scriptstyle f}\downarrow & & {\scriptstyle f^i}\downarrow & & {\scriptstyle f^{i+1}}\downarrow & & {\scriptstyle f^{i+2}}\downarrow & & \\
0 & \longrightarrow & \Sigma^{n+i} L & \longrightarrow & J^{n+i} & \longrightarrow & J^{n+i+1} & \longrightarrow & J^{n+i+2} & \longrightarrow & \dots
\end{array}$$

The restriction $c_i[(f^j)_{j \geq i}]$ is the cochain map $(f^j)_{j \geq i+1} : I^{\geq i+1} \longrightarrow J^*[-n]$ which lifts the linear map $g : \Sigma^{i+1} N \longrightarrow \Sigma^{i+n+1} L$ induced from f^i by passage to the quotients

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^i N & \longrightarrow & I^i & \longrightarrow & \Sigma^{i+1} N & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & {\scriptstyle f^i}\downarrow & & {\scriptstyle g}\downarrow & & \\
0 & \longrightarrow & \Sigma^{n+i} L & \longrightarrow & J^{n+i} & \longrightarrow & \Sigma^{n+i+1} L & \longrightarrow & 0
\end{array}$$

The proof is complete, since the linear map g represents the image in $\mathrm{Ext}_R^{n+i+1}(\Sigma^{i+1} N, L)$ of ξ under the connecting homomorphism

$$d_n^i : \mathrm{Ext}_R^{n+i}(\Sigma^i N, L) \longrightarrow \mathrm{Ext}_R^{n+i+1}(\Sigma^{i+1} N, L).$$

(iii) In view of equation (2), this is an immediate consequence of (ii) above, which identifies the direct system $\big(H^n\big(\mathrm{Hom}_R\big(I^{\geq i}, J^*\big)\big)\big)_i$, with structure maps given by the c_i^n's, with the direct system $\big(\mathrm{Ext}_R^{n+i}(\Sigma^i N, L)\big)_i$, with structure maps given by the d_i^n's. □

Remark 1.3. There is a similar description of Mislin's complete cohomology $\widehat{\mathrm{Ext}}$, in terms of a decreasing filtration of the total Hom complex, which is associated with projective resolutions of both variables; cf. [5, Lemma A.3].

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2. The relation between stable and complete homology

Let R be a ring and consider a right R-module M and a left R-module N. Vogel has defined in [12] the stable homology groups $\widetilde{\mathrm{Tor}}_*^R(M,N)$. If $P_* \longrightarrow M \longrightarrow 0$ is a projective resolution of M and $0 \longrightarrow N \longrightarrow I^*$ an injective resolution of N, then

$$\widetilde{\mathrm{Tor}}_n^R(M,N) = H_{n+1}\big(P_* \,\widetilde{\otimes}_R\, I^*\big)$$

for all n. Here, $P_* \,\widetilde{\otimes}_R\, I^*$ is the complex of abelian groups whose group of chains in degree n is the quotient of the direct product $\prod_{i-j=n} P_i \otimes_R I^j$ modulo the direct sum $\bigoplus_{i-j=n} P_i \otimes_R I^j$. The reader may consult [12] and [4] for several basic properties of the stable homology groups. We shall now recall some of these properties: The stable homology groups $\widetilde{\mathrm{Tor}}_*^R(M,N)$ may be computed by replacing the projective resolution $P_* \longrightarrow M \longrightarrow 0$ of M in the definition above by any flat resolution $F_* \longrightarrow M \longrightarrow 0$ of M; we have

$$\widetilde{\mathrm{Tor}}_n^R(M,N) = H_{n+1}\big(F_* \,\widetilde{\otimes}_R\, I^*\big)$$

for all n. Consequently, the groups $\widetilde{\mathrm{Tor}}_*^R(M,N)$ vanish if either M has finite flat dimension or else if N has finite injective dimension.

The complete homology groups $\widehat{\mathrm{Tor}}_*^R(M,N)$ were defined by Triulzi in [21] as the evaluation of the injective completion $\widehat{\mathrm{Tor}}_*^R(M,\_)$ of the ordinary Tor functor $\mathrm{Tor}_*^R(M,\_)$ to the left R-module N. These groups vanish if N is injective and are universal (in a certain sense) with that property. Triulzi's construction involves the notion of right satellites of functors, which is due to Cartan and Eilenberg [3, Chapter III]. There is a concrete description of complete homology, which may be expressed in terms of an injective resolution $0 \longrightarrow N \longrightarrow I^*$ of N. Indeed, given such a resolution, we may consider the corresponding cosyzygy modules $\Sigma^i N$, $i \geq 0$, and note that the short exact sequence

$$0 \longrightarrow \Sigma^i N \longrightarrow I^i \longrightarrow \Sigma^{i+1} N \longrightarrow 0$$

induces connecting homomorphisms

$$\delta_n^i : \mathrm{Tor}_{n+i+1}^R(M,\Sigma^{i+1}N) \longrightarrow \mathrm{Tor}_{n+i}^R(M,\Sigma^i N) \tag{3}$$

for all n,i with $i \geq 0$. We may consider the inverse system $(\mathrm{Tor}_{n+i}^R(M,\Sigma^i N))_i$ with structure maps given by the connecting homomorphisms δ_n^i as above; it turns out that

$$\widehat{\mathrm{Tor}}_n^R(M,N) = \varprojlim_i \mathrm{Tor}_{n+i}^R(M,\Sigma^i N), \tag{4}$$

whereas the canonical map $\widehat{\mathrm{Tor}}_n^R(M,N) \longrightarrow \mathrm{Tor}_n^R(M,N)$ is identified with the natural projection of the limit

$$\varprojlim_i \mathrm{Tor}_{n+i}^R(M,\Sigma^i N) \longrightarrow \mathrm{Tor}_{n+0}^R(M,\Sigma^0 N)$$

for all n (cf. [5, Lemma 1.7]).

In order to relate the stable homology groups $\widetilde{\mathrm{Tor}}_*^R(M,N)$ to the complete homology groups $\widehat{\mathrm{Tor}}_*^R(M,N)$, we shall use yet another description of the latter, which may be formulated in terms of a flat resolution $F_* \longrightarrow M \longrightarrow 0$ of M and an injective resolution $0 \longrightarrow N \longrightarrow I^*$ of N (cf. [4, §2.4]). As in the previous section, we consider for all $i \geq 0$ the subcomplex $I^{\geq i} \subseteq I^*$, which coincides with I^* in (cohomological) degrees $j \geq i$ and vanishes in degrees $j < i$. We recall that the inclusion $\Sigma^i N \hookrightarrow I^i$ induces a quasi-isomorphism $\Sigma^i N[i] \longrightarrow I^{\geq i}$ for all $i \geq 0$.

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Since $I^{\geq i+1}$ is a subcomplex of $I^{\geq i}$, it follows that $F_* \otimes_R I^{\geq i+1}$ is a subcomplex of $F_* \otimes_R I^{\geq i}$; we shall denote by γ^i the inclusion $F_* \otimes_R I^{\geq i+1} \hookrightarrow F_* \otimes_R I^{\geq i}$ for all $i \geq 0$. We also consider the additive maps $\gamma_n^i : H_n\big(F_* \otimes_R I^{\geq i+1}\big) \longrightarrow H_n\big(F_* \otimes_R I^{\geq i}\big)$, which are induced by the γ^i's in homology for all n, i with $i \geq 0$.

Proposition 2.1. *Let the notation be as above.*

(i) The quasi-isomorphism $\Sigma^i N[i] \longrightarrow I^{\geq i}$ induces an isomorphism of abelian groups

$$Tor_{n+i}^R(M, \Sigma^i N) \xrightarrow{\sim} H_n\big(F_* \otimes_R I^{\geq i}\big)$$

for all n, i with $i \geq 0$.

(ii) There is a commutative diagram of abelian groups

$$\begin{array}{ccc} Tor_{n+i+1}^R(M, \Sigma^{i+1} N) & \xrightarrow{\sim} & H_n\big(F_* \otimes_R I^{\geq i+1}\big) \\ {\scriptstyle \delta_n^i} \downarrow & & \downarrow {\scriptstyle \gamma_n^i} \\ Tor_{n+i}^R(M, \Sigma^i N) & \xrightarrow{\sim} & H_n\big(F_* \otimes_R I^{\geq i}\big) \end{array}$$

for all n, i with $i \geq 0$. Here, the horizontal isomorphisms are those defined in (i), whereas δ_n^i is the connecting homomorphism (3).

(iii) There is a natural isomorphism $\widehat{Tor}_n^R(M, N) \simeq \varprojlim_i H_n(F_* \otimes_R I^{\geq i})$ for all n.

Proof. The flat resolution $F_* \longrightarrow M \longrightarrow 0$ of the right R-module M induces, by applying the Pontryagin duality functor D, an injective resolution $0 \longrightarrow DM \longrightarrow DF_*$ of the left R-module DM.

(i) Proposition 1.1(i) implies that the quasi-isomorphism $\Sigma^i N[i] \longrightarrow I^{\geq i}$ induces an isomorphism of abelian groups

$$H^n\big(\mathrm{Hom}_R\big(I^{\geq i}, DF_*\big)\big) \xrightarrow{\sim} \mathrm{Ext}_R^{n+i}(\Sigma^i N, DM)$$

for all n, i with $i \geq 0$. In view of the standard Hom-tensor duality, the latter isomorphism is precisely that obtained by applying the Pontryagin duality functor D to the additive map

$$\mathrm{Tor}_{n+i}^R(M, \Sigma^i N) \longrightarrow H_n\big(F_* \otimes_R I^{\geq i}\big)$$

in the statement. Since $\mathbb{Q}/\mathbb{Z}$ is faithfully injective (as an abelian group), it follows that the latter map is bijective, as needed.

(ii) We argue as in (i) above: Since the abelian group $\mathbb{Q}/\mathbb{Z}$ is faithfully injective, it suffices to verify the commutativity of the diagram, which is obtained from that in the statement of (ii), by applying the Pontryagin duality functor D. Invoking the standard Hom-tensor duality, the latter diagram is identified with

$$\begin{array}{ccc} \mathrm{Ext}_R^{n+i+1}(\Sigma^{i+1} N, DM) & \xleftarrow{\sim} & H^n\big(\mathrm{Hom}_R\big(I^{\geq i+1}, DF_*\big)\big) \\ {\scriptstyle d_i^n} \uparrow & & \uparrow {\scriptstyle c_i^n} \\ \mathrm{Ext}_R^{n+i}(\Sigma^i N, DM) & \xleftarrow{\sim} & H^n\big(\mathrm{Hom}_R\big(I^{\geq i}, DF_*\big)\big) \end{array}$$

Here, the functor D maps the connecting homomorphism δ_n^i (resp. γ_n^i) onto the connecting homomorphism d_i^n (resp. onto c_i^n). The proof is completed, since the latter diagram is commutative, as shown in Proposition 1.1(ii).

(iii) In view of equation (4), this is an immediate consequence of (ii) above, which identifies the inverse system $\big(H_n\big(F_* \otimes_R I^{\geq i}\big)\big)_i$, with structure maps given by the γ_n^i's, with the inverse system $\big(\mathrm{Tor}_{n+i}^R(M, \Sigma^i N)\big)_i$, with structure maps given by the δ_n^i's. □

We are now ready to state and prove the following result, establishing the relation between stable and complete homology.

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Theorem 2.2. *Let R be a ring and consider a right R-module M and a left R-module N. Then, for all n there is a short exact sequence*

$$0 \longrightarrow \varprojlim_i{}^1 \, Tor^R_{n+i+1}(M, \Sigma^i N) \longrightarrow \widetilde{Tor}^R_n(M,N) \xrightarrow{\partial} \widehat{Tor}^R_n(M,N) \longrightarrow 0,$$

which is natural in both M and N. Here, $\Sigma^i N$, $i \geq 0$, are the cosyzygy modules of N that are associated with any injective resolution of it, whereas the structure maps of the inverse systems $\big(Tor^R_{n+i}(M, \Sigma^i N)\big)_i$ are the connecting homomorphisms δ^i_n in (3) for all n.

Proof. Let $F_* \longrightarrow M \longrightarrow 0$ be a flat resolution of M and $0 \longrightarrow N \longrightarrow I^*$ an injective resolution of N. We consider the subcomplexes $I^{\geq i} \subseteq I^*$, $i \geq 0$, which were defined above and note that these provide us with a decreasing filtration of I^*

$$I^* = I^{\geq 0} \supseteq I^{\geq 1} \supseteq I^{\geq 2} \supseteq \ldots \supseteq I^{\geq i} \supseteq \ldots .$$

We also consider the induced filtration of the complex $F_* \otimes_R I^*$

$$F_* \otimes_R I^* = F_* \otimes_R I^{\geq 0} \supseteq F_* \otimes_R I^{\geq 1} \supseteq F_* \otimes_R I^{\geq 2} \supseteq \ldots \supseteq F_* \otimes_R I^{\geq i} \supseteq \ldots .$$

Having fixed the degree n, the associated decreasing filtration of the abelian group of n-chains $(F_* \otimes_R I^*)_n = \bigoplus_j F_{j+n} \otimes_R I^j = \bigoplus_{j\geq 0} F_{j+n} \otimes_R I^j$ of the complex $F_* \otimes_R I^*$ is presented below

$$\textstyle\bigoplus_{j\geq 0} F_{j+n} \otimes_R I^j \supseteq \bigoplus_{j\geq 1} F_{j+n} \otimes_R I^j \supseteq \bigoplus_{j\geq 2} F_{j+n} \otimes_R I^j \supseteq \ldots \supseteq \bigoplus_{j\geq i} F_{j+n} \otimes_R I^j \supseteq \ldots .$$

It follows readily that this filtration of the complex $F_* \otimes_R I^*$ is Hausdorff (cf. Definition A.1 of the Appendix). We wish to identify the complex $\varprojlim_i{}^1 \big(F_* \otimes I^{\geq i}\big)$. As noted in the discussion preceding Lemma A.2, the abelian group of n-chains of the latter complex is the cokernel of the natural map

$$(F_* \otimes_R I^*)_n \longrightarrow \varprojlim_i \big[(F_* \otimes_R I^*)_n / \big(F_* \otimes_R I^{\geq i}\big)_n\big]. \tag{5}$$

Since $(F_* \otimes_R I^*)_n = \bigoplus_{j\geq 0} F_{j+n} \otimes_R I^j$ and $\big(F_* \otimes_R I^{\geq i}\big)_n = \bigoplus_{j\geq i} F_{j+n} \otimes_R I^j$, we conclude that

$$(F_* \otimes_R I^*)_n / \big(F_* \otimes_R I^{\geq i}\big)_n = \textstyle\bigoplus_{j=0}^{i-1} F_{j+n} \otimes_R I^j = \prod_{j=0}^{i-1} F_{j+n} \otimes_R I^j$$

for all n, i with $i \geq 0$. It follows easily from this that the group $\varprojlim_i \big[(F_* \otimes_R I^*)_n / \big(F_* \otimes_R I^{\geq i}\big)_n\big]$ is isomorphic with the direct product $\prod_{j\geq 0} F_{j+n} \otimes_R I^j$, in such a way that the map (5) above is identified with the natural inclusion

$$\textstyle\bigoplus_{j\geq 0} F_{j+n} \otimes_R I^j \hookrightarrow \prod_{j\geq 0} F_{j+n} \otimes_R I^j.$$

We therefore conclude that the complex $\varprojlim_i{}^1 \big(F_* \otimes I^{\geq i}\big)$ coincides with the complex $F_* \widetilde{\otimes}_R I^*$, that computes the stable homology groups $\widetilde{\mathrm{Tor}}^R_*(M,N)$. Hence, applying Corollary A.3 of the Appendix to the special case of the Hausdorff filtration of $F_* \otimes_R I^*$ given by the $F_* \otimes I^{\geq i}$'s, we obtain for all n a short exact sequence of abelian groups

$$0 \longrightarrow \varprojlim_i{}^1 H_{n+1}\big(F_* \otimes_R I^{\geq i}\big) \longrightarrow H_{n+1}\big(F_* \widetilde{\otimes}_R I^*\big) \xrightarrow{\partial} \varprojlim_i H_n\big(F_* \otimes_R I^{\geq i}\big) \longrightarrow 0.$$

Proposition 2.1(ii) implies that the inverse system $\big(H_n\big(F_* \otimes_R I^{\geq i}\big)\big)_i$, with structure maps given by the γ^i_n's, is isomorphic with the inverse system $\big(\mathrm{Tor}^R_{n+i}(M, \Sigma^i N)\big)_i$, with structure maps given by the δ^i_n's. Hence, the exact sequence above reduces to that in the statement of the Theorem and the proof is complete. □

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Remark 2.3. Keeping the same notation as above, we note that for all n the map

$$\partial : \widetilde{\mathrm{Tor}}_n^R(M,N) \longrightarrow \widehat{\mathrm{Tor}}_n^R(M,N),$$

i.e. the map

$$\partial : H_{n+1}\big(F_* \widetilde{\otimes}_R I^*\big) \longrightarrow \varprojlim_i H_n\big(F_* \otimes_R I^{\geq i}\big)$$

resulting from the short exact sequence in Corollary A.3 of the Appendix, coincides with that defined by Triulzi; cf. [21, Lemma 6.2.9]. An explicit description of this map can be also found in [5, §2.5].

3. Isomorphism criteria

Having proved Theorem 2.2, one may look for conditions implying that the natural surjection from stable homology to complete homology is bijective. An answer to this problem may have important consequences for the two homology theories. For example, if the functors $\widetilde{\mathrm{Tor}}_*^R(M,\_)$ and $\widehat{\mathrm{Tor}}_*^R(M,\_)$ are naturally isomorphic, then:

(i) stable homology has the universal property inherited by the fact that it is naturally isomorphic with the injective completion of the functor $\mathrm{Tor}_*^R(M,\_)$ and

(ii) complete homology is a homological δ-functor, since it is naturally isomorphic with a functor that is computed as the homology of a complex (cf. [4, §2.4 and §2.5]).

In the present section, we shall obtain two types of conditions under which the natural map from stable homology to complete homology is bijective. We should point out that the results presented in Propositions 3.1 and 3.3 below are not new; they have been already proved in [5]. Our goal is to provide alternative proofs for these results, by using the identification of the kernel of the canonical map $\partial : \widetilde{\mathrm{Tor}} \longrightarrow \widehat{\mathrm{Tor}}$. In other words, we shall approach the problem by looking for conditions that imply the vanishing of the $\varprojlim^1$-term in the short exact sequence of Theorem 2.2.

Grothendieck introduced in [13] a simple condition for an inverse system of abelian groups $(A_i)_i$ that implies the vanishing of $\varprojlim_i^1 A_i$, the so-called Mittag-Leffler condition. We say that the inverse system $(A_i)_i$ satisfies the Mittag-Leffler condition if the decreasing filtration induced on A_i by the images of the structure maps from A_j, $j \geq i$, is eventually constant for all i. In other words, we demand that for all i there exists $j(i) \geq i$, such that the image of the structure map $A_{j(i)} \longrightarrow A_i$ is equal to the image of the structure map $A_k \longrightarrow A_i$ for all $k \geq j(i)$; in that case, the image A'_i of the map $A_{j(i)} \longrightarrow A_i$ is referred to as the stable image. It is clear that the Mittag-Leffler condition is satisfied if the structure maps $A_i \longrightarrow A_{i-1}$ are surjective for all $i \gg 0$.

Proposition 3.1. *Let R be a ring and consider a right R-module M. We assume that one of the following two conditions is satisfied:*

(i) There exists an integer m, such that the functors $Tor_i^R(M,\_)$ vanish on all injective left R-modules for all $i > m$.[1]

(ii) All injective left R-modules have finite flat dimension.[2]

Then, the natural map $\partial : \widetilde{Tor}_*^R(M,N) \longrightarrow \widehat{Tor}_*^R(M,N)$ is bijective for all left R-modules N.

[1] In the terminology of [10], this condition says that the module M has finite copure flat dimension.

[2] In that case, there is an upper bound on the flat lengths of injective left R-modules; in the terminology of [9] (see also [6] and [15]), this condition says that the invariant $\mathrm{sfli}\, R$ is finite.

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Proof. Let $0 \longrightarrow N \longrightarrow I^*$ be an injective resolution of the left R-module N. Then, both assumptions (i) and (ii) imply that there exists an integer m, such that the groups $\mathrm{Tor}_i^R(M, I^j)$ vanish for all $i > m$ and all j. Denoting by $\Sigma^i N$, $i \geq 0$, the corresponding cosyzygy modules of N, the exact sequence of abelian groups

$$\mathrm{Tor}_{n+i+1}^R(M, \Sigma^{i+1}N) \xrightarrow{\delta_n^i} \mathrm{Tor}_{n+i}^R(M, \Sigma^i N) \longrightarrow \mathrm{Tor}_{n+i}^R(M, I^i),$$

which is induced by the short exact sequence of left R-modules

$$0 \longrightarrow \Sigma^i N \longrightarrow I^i \longrightarrow \Sigma^{i+1}N \longrightarrow 0,$$

implies that the connecting homomorphism

$$\delta_n^i : \mathrm{Tor}_{n+i+1}^R(M, \Sigma^{i+1}N) \longrightarrow \mathrm{Tor}_{n+i}^R(M, \Sigma^i N)$$

is surjective for all $i > m - n$. It follows from the preceding discussion that the inverse system $\big(\mathrm{Tor}_{n+i}^R(M, \Sigma^i N)\big)_i$ satisfies the Mittag-Leffler condition and hence $\varprojlim_i{}^1 \mathrm{Tor}_{n+i}^R(M, \Sigma^i N) = 0$ for all n. Invoking Theorem 2.2, this completes the proof. □

Another type of condition on an inverse system that implies the vanishing of the corresponding $\varprojlim^1$ is obtained by dualizing a direct system, by means of a contravariant exact functor which maps direct sums onto direct products. We shall illustrate this by considering the Pontryagin duality functor D. We note that any direct system $(B_i)_i$ of abelian groups with structure maps $\lambda_i : B_i \longrightarrow B_{i+1}$ induces an inverse system of abelian groups $(DB_i)_i$, with structure maps $D\lambda_i : DB_{i+1} \longrightarrow DB_i$ for all i.

Lemma 3.2. *Let $(B_i)_i$ be a direct system of abelian groups and consider the associated inverse system $(DB_i)_i$, as above. Then, $\varprojlim_i DB_i \simeq D\Big(\varinjlim_i B_i\Big)$ and $\varprojlim_i{}^1 DB_i = 0$.*

Proof. The colimit $B = \varinjlim_i B_i$ fits into a short exact sequence

$$0 \longrightarrow \textstyle\bigoplus_i B_i \xrightarrow{1-\lambda} \bigoplus_i B_i \longrightarrow B \longrightarrow 0,$$

where $1 - \lambda$ is the map $(b_i)_i \mapsto (b_i - \lambda_{i-1}b_{i-1})_i$, $(b_i)_i \in \bigoplus_i B_i$. Applying the duality functor D to the above sequence, we obtain the short exact sequence

$$0 \longrightarrow DB \longrightarrow \textstyle\prod_i DB_i \xrightarrow{1-D\lambda} \prod_i DB_i \longrightarrow 0.$$

It follows readily that $\varprojlim_i DB_i = DB$ and $\varprojlim_i{}^1 DB_i$ is the trivial group. □

If M, K are two right R-modules, then we may define the additive map

$$\Phi : M \otimes_R DK \longrightarrow D\mathrm{Hom}_R(M, K),$$

by letting $\Phi(m \otimes f)$ be the operator which is given by $g \mapsto f(g(m))$, $g \in \mathrm{Hom}_R(M, K)$, for all $m \in M$ and $f \in DK$. The map Φ is natural in M, K and has been introduced by Cartan and Eilenberg in [3, Chapter VI, §5]. It is bijective if M is finitely presented.

Having fixed a right R-module K, we consider for any right R-module M a projective resolution $P_* \longrightarrow M \longrightarrow 0$. Then, the natural transformation Φ induces a chain map

$$\Phi : P_* \otimes_R DK \longrightarrow D\mathrm{Hom}_R(P_*, K).$$

By applying homology, we obtain additive maps

$$\Phi_i : \mathrm{Tor}_i^R(M, DK) \longrightarrow D\mathrm{Ext}_R^i(M, K),$$

$i \geq 0$, which do not depend upon the particular choice of the projective resolution of M. The Φ_i's are natural in both M and K and commute with the connecting homomorphisms which are associated with any short exact sequence of right R-modules

$$0 \longrightarrow K' \longrightarrow K \longrightarrow K'' \longrightarrow 0.$$

If M is of type FP_∞, i.e. if M admits a projective resolution $P_* \longrightarrow M \longrightarrow 0$, which consists of finitely generated (projective) modules in each degree, then the Φ_i's are bijective for all i.

Proposition 3.3. *Let R be a ring and consider a right R-module M of type FP_∞. Then, the natural map $\partial : \widetilde{Tor}_*^R(M, DK) \longrightarrow \widehat{Tor}_*^R(M, DK)$ is bijective for any right R-module K.*

Proof. Let $F_* \longrightarrow K \longrightarrow 0$ be a flat resolution of the right R-module K and denote by K_i the image of the differential $F_i \longrightarrow F_{i-1}$ for all $i \geq 1$. Then, $0 \longrightarrow DK \longrightarrow DF_*$ is an injective resolution of the left R-module DK and DK_i is identified with the corresponding i-th cosyzygy module $\Sigma^i DK$ for all $i \geq 1$. It follows from the discussion above that the inverse system $\left(\mathrm{Tor}_{n+i}^R(M, DK_i)\right)_i$, whose structure maps are connecting homomorphisms induced by the short exact sequences

$$0 \longrightarrow DK_i \longrightarrow DF_i \longrightarrow DK_{i+1} \longrightarrow 0,$$

is isomorphic with the inverse system $\left(D\mathrm{Ext}_R^{n+i}(M, K_i)\right)_i$, which is induced by applying the duality functor D to the direct system $\left(\mathrm{Ext}_R^{n+i}(M, K_i)\right)_i$, whose structure maps are connecting homomorphisms induced by the short exact sequences

$$0 \longrightarrow K_{i+1} \longrightarrow F_i \longrightarrow K_i \longrightarrow 0.$$

It follows from Lemma 3.2 that

$$\varprojlim_i{}^1 \mathrm{Tor}_{n+i}^R(M, DK_i) = \varprojlim_i{}^1 D\mathrm{Ext}_R^{n+i}(M, K_i) = 0$$

for all n and hence the proof is finished by invoking Theorem 2.2. $\square$

Remark 3.4. Keeping the same notation as in the proof of Proposition 3.3, we note that the groups $\widetilde{\mathrm{Tor}}_n^R(M, DK) \simeq \widehat{\mathrm{Tor}}_n^R(M, DK)$ are isomorphic with the limit

$$\varprojlim_i \mathrm{Tor}_{n+i}^R(M, DK_i) = \varprojlim_i D\mathrm{Ext}_R^{n+i}(M, K_i).$$

As noted in Lemma 3.2, the latter limit is identified with the Pontryagin dual of the colimit $\varinjlim_i \mathrm{Ext}_R^{n+i}(M, K_i)$ of the direct system $\left(\mathrm{Ext}_R^{n+i}(M, K_i)\right)_i$. This identification is valid for any flat resolution $F_* \longrightarrow K \longrightarrow 0$ of the right R-module K. Assuming that the flat modules F_i are in fact projective (so that $F_* \longrightarrow K \longrightarrow 0$ is a projective resolution of K), it follows that $K_i = \Omega^i K$ is the i-th syzygy module of K for all $i \geq 1$. Then, as shown in [18, §4], the colimit

$$\varinjlim_i \mathrm{Ext}_R^{n+i}(M, K_i) = \varinjlim_i \mathrm{Ext}_R^{n+i}(M, \Omega^i K)$$

is isomorphic with Mislin's complete cohomology group $\widehat{\mathrm{Ext}}_R^n(M, K)$. The identification of the D-dual of that group with $\widetilde{\mathrm{Tor}}_n^R(M, DK) \simeq \widehat{\mathrm{Tor}}_n^R(M, DK)$ has been noted during the proof of [5, Theorem 2.14].

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4. Vanishing criteria

As another application of Theorem 2.2, we may study the vanishing of stable homology. It will turn out that duality considerations reveal an interesting connection between the vanishing of the stable homology functors $\widetilde{\mathrm{Tor}}$ and that of Nucinkis' complete cohomology functors $\widetilde{\mathrm{Ext}}$.

Let $(A_i)_i$ be an inverse system of abelian groups with structure maps $\tau_i : A_{i+1} \longrightarrow A_i$, $i \geq 0$. The duality functor D may be used in order to construct a direct system of abelian groups $(DA_i)_i$, with structure maps $D\tau_i : DA_i \longrightarrow DA_{i+1}$, $i \geq 0$. We say that the inverse system $(A_i)_i$ is essentially zero if for all i there exists $j(i) \geq i$, such that the structure map $A_{j(i)} \longrightarrow A_i$ is the zero map. The limit of an essentially zero inverse system is clearly trivial; moreover, since such a system satisfies the Mittag-Leffler condition, its $\varprojlim^1$ is also trivial.

Lemma 4.1. *Let $(A_i)_i$ be an inverse system of abelian groups. Then, the following conditions are equivalent:*

(i) $\varprojlim_i A_i^{(X)} = \varprojlim_i^1 A_i^{(X)} = 0$ *for any set X.*

(ii) $\varprojlim_i A_i = \varprojlim_i^1 A_i^{(\mathbb{N})} = 0$.

(iii) The inverse system $(A_i)_i$ is essentially zero.

(iv) $\varinjlim_i D\left(A_i^{(X)}\right) = 0$ *for any set X.*

Proof. It is clear that (i)→(ii). We shall complete the proof, by showing that (ii)→(iii)→(i) and (iii)↔(iv).

(ii)→(iii): As shown in [7], the triviality of the group $\varprojlim_i^1 A_i^{(\mathbb{N})}$ implies that $(A_i)_i$ satisfies the Mittag-Leffler condition. Let $(A'_i)_i$ be the inverse system of stable images of $(A_i)_i$; we note that the structure maps $A'_{i+1} \longrightarrow A'_i$ are surjective for all i. Since $(A'_i)_i$ is a subsystem of $(A_i)_i$, we have $\varprojlim_i A'_i \subseteq \varprojlim_i A_i$ and hence $\varprojlim_i A'_i = 0$. It follows readily that $A'_i = 0$ for all i and hence the inverse system $(A_i)_i$ is essentially zero, as needed.

(iii)→(i): Since $(A_i)_i$ is assumed to be essentially zero, the system $\left(A_i^{(X)}\right)_i$ is essentially zero for any set X. As we have noted above, this implies that $\varprojlim_i A_i^{(X)} = \varprojlim_i^1 A_i^{(X)} = 0$.

(iii)→(iv): In view of our assumption, for any i there exists $j(i) \geq i$, such that the structure map $A_{j(i)} \longrightarrow A_i$ is trivial. It follows that the map $A_{j(i)}^{(X)} \longrightarrow A_i^{(X)}$ is trivial for any set X. By applying the duality functor D, we conclude that the map $D\left(A_i^{(X)}\right) \longrightarrow D\left(A_{j(i)}^{(X)}\right)$ is trivial as well. It follows readily that $\varinjlim_i DA_i^{(X)} = 0$.

(iv)→(iii): Since the abelian group $\mathbb{Q}/\mathbb{Z}$ is faithfully injective, we may choose X, in such a way that A_i admits a monomorphism f_i into the direct product $\Delta = (\mathbb{Q}/\mathbb{Z})^X$ for all i.[3] We note that the direct system $\left(D\left(A_i^{(X)}\right)\right)_i$ may be identified with the direct system $(\mathrm{Hom}(A_i, \Delta))_i$, which is obtained from $(A_i)_i$ by applying the contravariant functor $\mathrm{Hom}(\_, \Delta)$. Having fixed i, the vanishing of the colimit $\varinjlim_i \mathrm{Hom}(A_i, \Delta) = \varinjlim_i D\left(A_i^{(X)}\right)$ implies that the image of the element $f_i \in \mathrm{Hom}(A_i, \Delta)$ must vanish in the group $\mathrm{Hom}(A_j, \Delta)$ for a suitable $j \geq i$; in other words, the composition $A_j \longrightarrow A_i \xrightarrow{f_i} \Delta$ must be the zero map. Since f_i is injective, this can

[3]Any abelian group A may be embedded into the direct product of $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ copies of $\mathbb{Q}/\mathbb{Z}$. Hence, X may be chosen to be any set of cardinality exceeding $\sum_i \mathrm{card}\,\mathrm{Hom}(A_i, \mathbb{Q}/\mathbb{Z})$.

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be true only if the structure map $A_j \longrightarrow A_i$ is already zero. Therefore, we have proved that the inverse system $(A_i)_i$ is essentially zero, as needed. □

Proposition 4.2. *Let R be a ring and consider a right R-module M and a left R-module N. If $\widetilde{Tor}_n^R(M,N) = \widetilde{Tor}_{n-1}^R\big(M^{(\mathbb{N})},N\big) = 0$ for some n, then $\widetilde{Ext}_R^n(N,DM) = 0$.*

Proof. We fix an injective resolution $0 \longrightarrow N \longrightarrow I^*$ of the left R-module N and consider the corresponding cosyzygy modules $\Sigma^i N$, $i \geq 0$. We also consider the inverse system of abelian groups $(A_i)_i$, with $A_i = \mathrm{Tor}_{n+i}^R(M,\Sigma^i N)$ for all $i \geq 0$ and structure maps given by connecting homomorphisms which are induced by the short exact sequences

$$0 \longrightarrow \Sigma^i N \longrightarrow I^i \longrightarrow \Sigma^{i+1} N \longrightarrow 0.$$

In view of Theorem 2.2, the vanishing of the groups $\widetilde{\mathrm{Tor}}_n^R(M,N)$ and $\widetilde{\mathrm{Tor}}_{n-1}^R\big(M^{(\mathbb{N})},N\big)$ implies that $\varprojlim_i A_i = \varprojlim_i{}^1 A_i^{(\mathbb{N})} = 0$. On the other hand, the group $\widetilde{\mathrm{Ext}}_R^n(N,DM)$ is isomorphic with the colimit of the direct system $\big(\mathrm{Ext}_R^{n+i}(\Sigma^i N,DM)\big)_i$, whose structure maps are connecting homomorphisms which are induced by the short exact sequences above. Since the latter system may be naturally identified with the direct system $(DA_i)_i$, we may invoke Lemma 4.1 in order to conclude that $\widetilde{\mathrm{Ext}}_R^n(N,DM) = \varinjlim_i \mathrm{Ext}_R^{n+i}(\Sigma^i N,DM) = \varinjlim_i DA_i = 0$. □

Corollary 4.3. *Let R be a ring and consider a right R-module M. Then, the following conditions are equivalent:*

(i) $\widetilde{Tor}_*^R(\_,DM) = 0$,

(ii) $\widetilde{Tor}_0^R(M,DM) = \widetilde{Tor}_{-1}^R\big(M^{(\mathbb{N})},DM\big) = 0$ and

(iii) $\mathrm{fd}_{R^o} M < \infty$.

Proof. (i)→(ii): This is obvious.

(ii)→(iii): The vanishing of the stable homology groups implies, in view of Proposition 4.2, that $\widetilde{\mathrm{Ext}}_R^0(DM,DM) = 0$ and hence $\mathrm{id}_R DM < \infty$. The result follows since $\mathrm{fd}_{R^o} M = \mathrm{id}_R DM$.

(iii)→(i): Since $\mathrm{id}_R DM = \mathrm{fd}_{R^o} M < \infty$, the stable homology functors $\widetilde{\mathrm{Tor}}_*^R(\_,DM)$ vanish identically. □

Proposition 4.4. *Let R be a ring and consider a right R-module M and a left R-module N. Then, the following conditions are equivalent:*

(i) $\widetilde{Tor}_*^R\big(M^{(X)},N\big) = 0$ for any set X,

(ii) $\widetilde{Tor}_*^R\big(M^{(\mathbb{N})},N\big) = 0$ and

(iii) $\widetilde{Ext}_R^*\big(N,D\left(M^{(X)}\right)\big) = 0$ for any set X.

Proof. The proof proceeds along the same lines as that of Proposition 4.2. We fix an injective resolution $0 \longrightarrow N \longrightarrow I^*$ of the left R-module N and consider the corresponding cosyzygy modules $\Sigma^i N$, $i \geq 0$, and the inverse system of abelian groups $\big(\mathrm{Tor}_{n+i}^R(M,\Sigma^i N)\big)_i$ for all n. Then, Theorem 2.2 implies that conditions (i) and (ii) in the statement are equivalent to the following two conditions respectively:

(a) $\varprojlim_i \mathrm{Tor}_{n+i}^R(M,\Sigma^i N)^{(X)} = \varprojlim_i{}^1 \mathrm{Tor}_{n+i}^R(M,\Sigma^i N)^{(X)} = 0$ for all n and any set X,

(b) $\varprojlim_i \mathrm{Tor}_{n+i}^R(M,\Sigma^i N)^{(\mathbb{N})} = \varprojlim_i{}^1 \mathrm{Tor}_{n+i}^R(M,\Sigma^i N)^{(\mathbb{N})} = 0$ for all n.

On the other hand, for any set X the direct system $\big(\mathrm{Ext}_R^{n+i}\big(\Sigma^i N, D\left(M^{(X)}\right)\big)\big)_i$, whose colimit is the complete cohomology group $\widetilde{\mathrm{Ext}}_R^n\big(N,D\left(M^{(X)}\right)\big)$, may be naturally identified with the induced direct system $\big(D\big(\mathrm{Tor}^R_{n+i}(M,\Sigma^i N)^{(X)}\big)\big)_i$. Hence, condition (iii) in the statement is equivalent to

(c) $\varinjlim_i D\big(\mathrm{Tor}^R_{n+i}(M,\Sigma^i N)^{(X)}\big) = 0$ for all n and any set X.

It follows that the equivalence of the three conditions in the statement is an immediate consequence of Lemma 4.1. □

Corollary 4.5. *Let R be a ring and consider a left R-module N. Then, the following conditions are equivalent:*

(i) $\widetilde{Tor}^R_*(\_,N) = 0$ *and*

(ii) $\widetilde{Ext}^*_R(N,D\_) = 0$. □

5. Vanishing criteria over Noetherian rings

In this section, we shall restrict our attention to the case where the ring R is left Noetherian, in order to obtain clean characterizations of modules for which the partial stable homology functors vanish. As we have already noted before, it is an immediate consequence of the definition of stable homology that the functors $\widetilde{\mathrm{Tor}}^R_*(M,\_)$ vanish identically if M is a right R-module of finite flat dimension, whereas the functors $\widetilde{\mathrm{Tor}}^R_*(\_,N)$ vanish identically if N is a left R-module of finite injective dimension. We shall prove that both of these implications may be reversed, in the case where R is left Noetherian.

We begin with the functors $\widetilde{\mathrm{Tor}}^R_*(M,\_)$, where M is a right R-module. In order to examine their vanishing, we shall need the following variants of Propositions 4.2 and 4.4.

Proposition 5.1. *Let R be a left Noetherian ring and consider a right R-module M and a left R-module N. If* $\widetilde{Tor}^R_n(M,N) = \widetilde{Tor}^R_{n-1}\big(M,N^{(\mathbb{N})}\big) = 0$ *for some n, then* $\widetilde{Ext}^n_R(N,DM) = 0$.

Proof. We fix an injective resolution $0 \longrightarrow N \longrightarrow I^*$ of the left R-module N and consider the corresponding cosyzygy modules $\Sigma^i N$, $i \geq 0$. In view of our hypothesis on R, the direct sum of injective left R-modules is also injective (cf. [1]). Therefore, $0 \longrightarrow N^{(\mathbb{N})} \longrightarrow I^{*(\mathbb{N})}$ is an injective resolution of the left R-module $N^{(\mathbb{N})}$; the corresponding cosyzygy modules $\Sigma^i\big(N^{(\mathbb{N})}\big)$ are thereby identified with $(\Sigma^i N)^{(\mathbb{N})}$ for all $i \geq 0$.

From this point on, the proof proceeds as that of Proposition 4.2. We consider the inverse system of abelian groups $(A_i)_i$ with $A_i = \mathrm{Tor}^R_{n+i}(M,\Sigma^i N)$ for all $i \geq 0$ and structure maps given by connecting homomorphisms which are induced by the short exact sequences

$$0 \longrightarrow \Sigma^i N \longrightarrow I^i \longrightarrow \Sigma^{i+1} N \longrightarrow 0.$$

In view of Theorem 2.2, the vanishing of the groups $\widetilde{\mathrm{Tor}}^R_n(M,N)$ and $\widetilde{\mathrm{Tor}}^R_{n-1}\big(M,N^{(\mathbb{N})}\big)$ implies that $\varprojlim_i A_i = \varprojlim_i{}^1 A_i^{(\mathbb{N})} = 0$. On the other hand, the group $\widetilde{\mathrm{Ext}}^n_R(N,DM)$ is isomorphic with the colimit of the direct system $\big(\mathrm{Ext}^{n+i}_R(\Sigma^i N,DM)\big)_i$, whose structure maps are connecting homomorphisms which are induced by the short exact sequences above. Since the latter system may be naturally identified with the induced direct system $(DA_i)_i$, we may invoke Lemma 4.1 in order to conclude that $\widetilde{\mathrm{Ext}}^n_R(N,DM) = \varinjlim_i \mathrm{Ext}^{n+i}_R(\Sigma^i N,DM) = \varinjlim_i DA_i = 0$. □

Proposition 5.2. *Let R be a left Noetherian ring and consider a right R-module M and a left R-module N. Then, the following conditions are equivalent:*

(i) $\widetilde{Tor}^R_*\big(M,N^{(X)}\big) = 0$ *for any set X,*

(ii) $\widetilde{Tor}_*^R\big(M, N^{(\mathbb{N})}\big) = 0$ *and*

(iii) $\widetilde{Ext}_R^*\big(N^{(X)}, DM\big) = 0$ *for any set* X.

Proof. We fix an injective resolution $0 \longrightarrow N \longrightarrow I^*$ of the left R-module N and consider the corresponding cosyzygy modules $\Sigma^i N$, $i \geq 0$. As in the proof of Proposition 5.1, our hypothesis on R implies that $0 \longrightarrow N^{(X)} \longrightarrow I^{*(X)}$ is an injective resolution of the left R-module $N^{(X)}$ for any set X and hence the corresponding cosyzygy modules $\Sigma^i\big(N^{(X)}\big)$ are identified with $(\Sigma^i N)^{(X)}$ for all $i \geq 0$.

From this point on, the proof proceeds as that of Proposition 4.4. Invoking Theorem 2.2, we conclude that conditions (i) and (ii) in the statement are equivalent to the following two conditions respectively:

(a) $\varprojlim_i \mathrm{Tor}_{n+i}^R(M, \Sigma^i N)^{(X)} = \varprojlim_i^1 \mathrm{Tor}_{n+i}^R(M, \Sigma^i N)^{(X)} = 0$ for all n and any set X.

(b) $\varprojlim_i \mathrm{Tor}_{n+i}^R(M, \Sigma^i N)^{(\mathbb{N})} = \varprojlim_i^1 \mathrm{Tor}_{n+i}^R(M, \Sigma^i N)^{(\mathbb{N})} = 0$ for all n.

On the other hand, for any set X the direct system $\big(\mathrm{Ext}_R^{n+i}\big((\Sigma^i N)^{(X)}, DM\big)\big)_i$, whose colimit is the complete cohomology group $\widetilde{\mathrm{Ext}}_R^n\big(N^{(X)}, DM\big)$, may be naturally identified with the induced direct system $\big(D\big(\mathrm{Tor}_{n+i}^R(M, \Sigma^i N)^{(X)}\big)\big)_i$. Hence, condition (iii) in the statement is equivalent to

(c) $\varinjlim_i D\big(\mathrm{Tor}_{n+i}^R(M, \Sigma^i N)^{(X)}\big) = 0$ for all n and any set X.

It follows that the equivalence of the three conditions in the statement is an immediate consequence of Lemma 4.1. □

We can now state and prove the following result, which complements Corollary 4.3 (under the presence of the additional assumption that R is left Noetherian).

Theorem 5.3. *Let R be a left Noetherian ring. Then, the following conditions are equivalent for a right R-module M:*

(i) $\widetilde{Tor}_*^R(M, \_) = 0$,

(i') $\widetilde{Tor}_0^R(M, DM) = \widetilde{Tor}_{-1}^R\big(M, (DM)^{(\mathbb{N})}\big) = 0$,

(ii) $\widetilde{Tor}_*^R(\_, DM) = 0$,

(ii') $\widetilde{Tor}_0^R(M, DM) = \widetilde{Tor}_{-1}^R\big(M^{(\mathbb{N})}, DM\big) = 0$,

(iii) $\widetilde{Ext}_R^*(\_, DM) = 0$ *and*

(iv) $fd_{R^o} M < \infty$.

Proof. The equivalence between conditions (ii), (ii') and (iv) was established in Corollary 4.3 (without any Noetherian assumption on R), whereas the equivalence between (i) and (iii) follows from Proposition 5.2. Of course, it is obvious that (i)→(i') and he have already noted that the implication (iv)→(i) is an immediate consequence of the definition of stable homology (without any Noetherian assumption on R). Therefore, it only remains to prove that (i')→(iv). In order to prove the latter implication, assume that (i') holds. Then, Proposition 5.1 implies that $\widetilde{\mathrm{Ext}}_R^0(DM, DM) = 0$ and hence $\mathrm{fd}_{R^o} M = \mathrm{id}_R DM < \infty$, as needed. □

We shall now turn our attention to the case of the stable homology functors $\widetilde{\mathrm{Tor}}_*^R(\_, N)$, where N is a left R-module. In order to examine whether their vanishing implies the finiteness of $\mathrm{id}_R N$, we shall employ an injectivity criterion, which involves the notion of purity. We recall

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that a short exact sequence of left R-modules

$$0 \longrightarrow L' \xrightarrow{i} L \longrightarrow L'' \longrightarrow 0$$

is called pure if the induced sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(C, L') \longrightarrow \mathrm{Hom}_R(C, L) \longrightarrow \mathrm{Hom}_R(C, L'') \longrightarrow 0$$

is exact for any finitely presented left R-module C. In that case, we also say that i is a pure monomorphism. The condition defining purity is easily seen to be equivalent to the following assertion: If $f : A \longrightarrow B$ is a linear map whose cokernel $\operatorname{coker} f$ is finitely presented and

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ L' & \xrightarrow{i} & L \end{array}$$

is a commutative diagram of left R-modules, then there exists a linear map $g : B \longrightarrow L'$ with $a = gf$. We note that for any left R-module L the natural map $\nu_L : L \longrightarrow D^2L$ is a pure monomorphism; for a proof of that assertion, the reader may consult [20, §II.1.1.5].

Lemma 5.4. *Let R be a left Noetherian ring. Then, the following conditions are equivalent for a left R-module L:*
(i) L is injective,
(ii) the natural map $\nu_L : L \longrightarrow D^2L$ factors through an injective module and
(iii) there is a pure monomorphism $i : L \longrightarrow N$, which factors through an injective module.

Proof. Since the implications (i)→(ii)→(iii) are obvious, it only remains to prove that (iii)→(i). To that end, we fix a pure monomorphism $i : L \longrightarrow N$ that factors as the composition $L \xrightarrow{a} I \xrightarrow{b} N$, where I is an injective module. We shall prove that L is injective, using Baer's criterion: We consider a left ideal $\mathfrak{A} \subseteq R$ and a linear map $f : \mathfrak{A} \longrightarrow L$. Since I is injective, the composition $af : \mathfrak{A} \longrightarrow I$ extends to a linear map $g : R \longrightarrow I$. We now consider the following diagram

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{j} & R \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle bg} \\ L & \xrightarrow{i} & N \end{array}$$

where $j : \mathfrak{A} \longrightarrow R$ is the inclusion map. We note that this diagram is commutative, since $if = (ba)f = b(af) = b(gj) = (bg)j$. The ring R being left Noetherian, the left ideal $\mathfrak{A}$ is finitely generated and hence the cokernel $\operatorname{coker} j = R/\mathfrak{A}$ is finitely presented. Therefore, the purity of i implies the existence of a linear map $h : R \longrightarrow L$, such that $f = hj$, as needed. □

We can now state and prove the following result, which complements Corollary 4.5 (under the presence of the additional assumption that R is left Noetherian).

Theorem 5.5. *Let R be a left Noetherian ring. Then, the following conditions are equivalent for a left R-module N:*
(i) $\widetilde{Tor}^R_*(\_, N) = 0$,
(i') $\widetilde{Tor}^R_0(DN, N) = \widetilde{Tor}^R_{-1}\big((DN)^{(\mathbb{N})}, N\big) = 0$,
(ii) $\widetilde{Ext}^*_R(N, D\_) = 0$,
(ii') $\widetilde{Ext}^0_R(N, D^2N) = 0$ and
(iii) $id_R N < \infty$.

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Proof. The equivalence between conditions (i) and (ii) was established in Corollary 4.5. The implication (i)→(i') is obvious, whereas Proposition 4.2 implies that (i')→(ii'). We have already pointed out that the implication (iii)→(i) is an immediate consequence of the definition of stable homology. All of the above are valid without any Noetherian assumption on the ring R. We shall complete the proof by showing that (ii')→(iii); it is for this implication that we shall use our assumption that R is left Noetherian.

Let us therefore assume that $\widetilde{\mathrm{Ext}}^0_R(N, D^2N) = 0$ and proceed to show that $\mathrm{id}_R N$ is finite. We consider an injective resolution $0 \longrightarrow N \longrightarrow I^*$ of N and the corresponding cosyzygy modules $\Sigma^i N$, $i \geq 0$. Being left Noetherian, the ring R is left coherent; therefore, it follows that $DI^* \longrightarrow DN \longrightarrow 0$ is a flat resolution of the right R-module DN. Applying the functor D once more, it follows that $0 \longrightarrow D^2N \longrightarrow D^2I^*$ is an injective resolution of the left R-module D^2N. The corresponding cosyzygy modules $\Sigma^i(D^2N)$ are identified with the modules $D^2(\Sigma^i N)$ for all $i \geq 0$. Let $\nu_N : N \longrightarrow D^2N$ be the natural map; invoking the naturality of ν, we may lift ν_N to the cochain map $\nu_{I^*} : I^* \longrightarrow D^2I^*$. In this way, the linear map which is induced by ν_N between the corresponding cosyzygy modules is identified with the natural map $\nu_{\Sigma^i N} : \Sigma^i N \longrightarrow D^2(\Sigma^i N)$ for all $i \geq 0$. Since the complete cohomology group $\widetilde{\mathrm{Ext}}^0_R(N, D^2N)$ is trivial, the image of ν_N under the canonical map $\mathrm{Hom}_R(N, D^2N) \longrightarrow \widetilde{\mathrm{Ext}}^0_R(N, D^2N)$ must necessarily vanish. Hence, using the approach to complete cohomology by Benson and Carlson, we may conclude that the linear map which is induced by ν_N between the corresponding cosyzygy modules factors through an injective module in sufficiently large degrees. In other words, the linear map $\nu_{\Sigma^i N} : \Sigma^i N \longrightarrow D^2(\Sigma^i N)$ factors through an injective module for all $i \gg 0$. Since R is left Noetherian, Lemma 5.4 implies that $\Sigma^i N$ is an injective module for all $i \gg 0$; it follows that N has finite injective dimension, as needed. □

Remark 5.6. If R is an Artin algebra and N a finitely generated left R-module, the equivalence between conditions (i) and (iii) in Theorem 5.5 was proved in [4, Proposition 5.1].

6. Balance of stable homology

The definition of the stable homology groups $\widetilde{\mathrm{Tor}}^R_*(M, N)$, where M is a right R-module and N is a left R-module, has a built-in asymmetry, as it involves a flat resolution of M and an injective resolution of N. We could have used an injective resolution of M and a flat resolution of N, in order to define the stable homology groups $\widetilde{\mathrm{Tor}}^{R^o}_*(N, M)$. There is no a priori reason for the latter groups to coincide with the former ones; we say that stable homology is balanced for the pair (M, N) if the groups $\widetilde{\mathrm{Tor}}^R_n(M, N)$ and $\widetilde{\mathrm{Tor}}^{R^o}_n(N, M)$ are isomorphic for all n.

An analogous lack of symmetry is also present in the definition of the complete cohomology groups. The theory introduced by Mislin, which is based on projective modules (projective completions of functors, projective resolutions of modules), is not necessarily isomorphic with the dual theory, which was introduced by Nucinkis and is itself based on injective modules (injective completions of functors, injective resolutions of modules). In fact, Nucinkis has proved in [19, Theorem 5.2] that the two complete cohomology theories agree for all pairs of left R-modules if and only if all injective (resp. projective) left R-modules have finite projective (resp. injective) dimension. If this is the case, Gedrich and Gruenberg have shown in [11] that the invariants $\mathrm{silp}R$, the supremum of the injective lengths of projective left R-modules, and $\mathrm{spli}R$, the supremum of the projective lengths of injective left R-modules, are both finite and equal to each other.

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Our goal in this section is to examine the homological counterpart of the above result, describing conditions under which stable homology is balanced for any pair of (left and right) coefficient modules. If this is the case, we say that stable homology is balanced over R. It appears that this problem is closely related to the finiteness of the flat dimension of injective modules. If R is a ring, such that all injective left and all injective right R-modules have finite flat dimension, then the invariants sfli$R$ and sfliR^o, the suprema of the flat lengths of injective left and right R-modules respectively, are both finite and equal to each other; for a proof, see [9, Corollary 2.5].

A Tate flat resolution of the left R-module N is an acyclic complex of flat left R-modules, which remains acyclic upon tensoring with any injective right R-module and coincides with a flat resolution of N in sufficiently large degrees. Tate flat resolutions of right R-modules are defined analogously.

Proposition 6.1. *Let R be a ring and assume that the invariants sfli$R$ and sfliR^o are both finite. Then, stable homology is balanced over R.*

Proof. Assume that $\mathrm{sfli}\, R = n$. Then, as shown in [8, Lemma 5.2], for any left R-module N there exists an acyclic complex of flat left R-modules T_*, which coincides with a projective resolution of N in degrees $\geq n$. The finiteness of $\mathrm{sfli}\, R^o$ implies that T_* remains acyclic upon tensoring with any injective right R-module; therefore, T_* is a Tate flat resolution of N. A symmetric argument shows that any right R-module M admits a Tate flat resolution as well. Then, the balance of stable homology for the pair (M, N) follows from [4, Theorem 4.2]. □

We wish to prove that the sufficient condition for balance in Proposition 6.1 above is also necessary, i.e. that stable homology is balanced over R only if all injective left and all injective right R-modules have finite flat dimension. We shall prove below that this is indeed the case if the ring R is coherent on both sides.

Proposition 6.2. *Assume that stable homology is balanced over a ring R.*

(i) If R is either left Noetherian or right coherent, then $\mathrm{sfli}\, R^o < \infty$.

(ii) If R is either right Noetherian or left coherent, then $\mathrm{sfli}\, R < \infty$.

Proof. Since the balance of stable homology over a ring is a left-right symmetric condition, it only suffices to prove assertion (i). To that end, we consider an injective right R-module M and aim at proving that it has finite flat dimension.

The injectivity of M implies that the functors $\widetilde{\mathrm{Tor}}_*^R(M, \_) \simeq \widetilde{\mathrm{Tor}}_*^{R^o}(\_, M)$ vanish identically. If R is left Noetherian, then Theorem 5.3 implies that $\mathrm{fd}_{R^o} M < \infty$.

Assuming that R is right coherent, the left R-module DM is flat. Then, the stable homology functors $\widetilde{\mathrm{Tor}}_*^R(\_, DM) \simeq \widetilde{\mathrm{Tor}}_*^{R^o}(DM, \_)$ vanish identically and hence the finiteness of the flat dimension of M follows invoking Corollary 4.3. □

Theorem 6.3. *Let R be a ring, which is both left and right coherent. Then, the following conditions are equivalent:*

(i) stable homology is balanced over R and

(ii) $\mathrm{sfli}\, R = \mathrm{sfli}\, R^o < \infty$.

Proof. The implication (i)→(ii) follows from Proposition 6.2, whereas the implication (ii)→(i), which is valid over any ring, is proved in Proposition 6.1. □

Let R be a left Noetherian ring. Then, as shown by Iwanaga in [14], the left self-injective dimension of R (i.e. the injective dimension of the left regular module R) is equal to $\mathrm{sfli}\, R^o$.

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We say that a ring R is Iwanaga-Gorenstein if R is left and right Noetherian and has finite left and right self-injective dimension.

Corollary 6.4. *Let R be a ring, which is both left and right Noetherian. Then, the following conditions are equivalent:*
(i) stable homology is balanced over R and
(ii) R is Iwanaga-Gorenstein. $\square$

Remarks 6.5. (i) For an Artin algebra R, Corollary 6.4 was proved in [4, Corollary 4.5].
(ii) Let R be a ring over which stable homology is balanced and consider a right R-module M, for which either one of the following two conditions is satisfied:
(a) the left R-module DM has finite flat dimension or
(b) the right R-module $M^{(\mathbb{N})}$ has finite injective dimension.
Then, $\mathrm{fd}_{R^o} M < \infty$.
Indeed, assumption (a) implies that the functors $\widetilde{\mathrm{Tor}}_*^R(\_, DM) \simeq \widetilde{\mathrm{Tor}}_*^{R^o}(DM, \_)$ vanish identically and hence Corollary 4.3 implies that M has finite flat dimension. On the other hand, assumption (b) implies that the functors $\widetilde{\mathrm{Tor}}_*^R\left(M^{(\mathbb{N})}, \_\right) \simeq \widetilde{\mathrm{Tor}}_*^{R^o}\left(\_, M^{(\mathbb{N})}\right)$ vanish identically. Hence, Proposition 4.4 implies that the functors $\widetilde{\mathrm{Ext}}_R^*(\_, DM)$ vanish identically. In particular, $\widetilde{\mathrm{Ext}}_R^0(DM, DM) = 0$ and hence $\mathrm{fd}_{R^o} M = \mathrm{id}_R DM < \infty$.

Appendix A. Hausdorff filtrations on complexes and homology

We consider an inverse system of complexes of abelian groups $(X^i)_i$ with structure chain maps denoted by $\tau^i : X^{i+1} \longrightarrow X^i$ for all $i \geq 0$. Let

$$1 - \tau : \textstyle\prod_i X^i \longrightarrow \prod_i X^i$$

be the chain map, whose component in degree n is the additive map $\prod_i X_n^i \longrightarrow \prod_i X_n^i$, which is given by $(x^i)_i \mapsto (x^i - \tau^i x^{i+1})_i$, $(x_i)_i \in \prod_i X_n^i$. Then, the complexes $\varprojlim_i X^i$ and $\varprojlim_i^1 X^i$ are defined by means of the exact sequence

$$0 \longrightarrow \varprojlim_i X^i \longrightarrow \textstyle\prod_i X^i \xrightarrow{1-\tau} \prod_i X^i \longrightarrow \varprojlim_i^1 X^i \longrightarrow 0.$$

We are interested in the special case where $\varprojlim_i X^i$ is the zero complex; in other words, we assume that $\varprojlim_i X_n^i = 0$ for all n. Then, we obtain a short exact sequence of complexes

$$0 \longrightarrow \textstyle\prod_i X^i \xrightarrow{1-\tau} \prod_i X^i \longrightarrow \varprojlim_i^1 X^i \longrightarrow 0,$$

which induces a long exact sequence in homology

$$\ldots \longrightarrow H_{n+1}\left(\varprojlim_i^1 X^i\right) \xrightarrow{\partial} \textstyle\prod_i H_n(X^i) \xrightarrow{1-\tau} \prod_i H_n(X^i) \longrightarrow H_n\left(\varprojlim_i^1 X^i\right) \longrightarrow \ldots$$

Identifying the kernel and the cokernel of the additive map $1 - \tau : \prod_i H_n(X^i) \longrightarrow \prod_i H_n(X^i)$ as the $\varprojlim$ and $\varprojlim_i^1$ of the inverse system of abelian groups $(H_n(X^i))_i$ respectively, we obtain for all n a short exact sequence of abelian groups

$$0 \longrightarrow \varprojlim_i^1 H_{n+1}(X^i) \longrightarrow H_{n+1}\left(\varprojlim_i^1 X^i\right) \xrightarrow{\partial} \varprojlim_i H_n(X^i) \longrightarrow 0.$$

One source of examples of inverse systems of complexes of abelian groups with a vanishing limit, is obtained by considering complexes endowed with a Hausdorff filtration, in the sense of the following definition.

Definition A.1. *A Hausdorff filtration of a complex of abelian groups X is a decreasing sequence of subcomplexes $X = X^0 \supseteq X^1 \supseteq X^2 \supseteq \ldots \supseteq X^i \supseteq \ldots$, such that $\bigcap_i X^i = 0$ (i.e., such that $\bigcap_i X^i_n = 0$ for all n).*

Let X be a complex of abelian groups endowed with a Hausdorff filtration $(X^i)_i$ as above. Then, we may consider $(X^i)_i$ as an inverse system of complexes, with structure chain maps the inclusions $X^{i+1} \hookrightarrow X^i$ for all $i \geq 0$; in view of our assumption on the filtration, we have $\varprojlim_i X^i = \bigcap_i X^i = 0$. In order to identify $\varprojlim_i{}^1 X^i$, we consider the inverse system of complexes $(X/X^i)_i$ with structure chain maps $X/X^{i+1} \longrightarrow X/X^i$ induced by the identity of X for all $i \geq 0$. The projections $\pi^i : X \longrightarrow X/X^i$, $i \geq 0$, induce a chain map

$$\pi : X \longrightarrow \varprojlim_i X/X^i$$

and we claim that the complex $\varprojlim_i{}^1 X^i$ is isomorphic with the cokernel of π. In other words, we claim that there is an exact sequence of complexes

$$X \xrightarrow{\pi} \varprojlim_i X/X^i \longrightarrow \varprojlim_i{}^1 X^i \longrightarrow 0. \tag{6}$$

Since kernels, cokernels and products of complexes are computed degreewise, it suffices to show that for any degree n there is an exact sequence of abelian groups

$$X_n \xrightarrow{\pi} \varprojlim_i X_n/X^i_n \longrightarrow \varprojlim_i{}^1 X^i_n \longrightarrow 0.$$

Since $X_n = X^0_n \supseteq X^1_n \supseteq X^2_n \supseteq \ldots \supseteq X^i_n \supseteq \ldots$ is a filtration of the abelian group X_n, the identification of $\varprojlim_i{}^1 X^i_n$ claimed above is a consequence of the following folklore result.

Lemma A.2. *Let A be an abelian group, which is endowed with a decreasing filtration by subgroups $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots \supseteq A_i \supseteq \ldots$. Then, $\varprojlim_i{}^1 A_i$ is the cokernel of the canonical map $A \longrightarrow \varprojlim_i A/A_i$.*

Proof. This follows from the 6-term $\varprojlim$-$\varprojlim^1$ exact sequence, which is induced from the short exact sequence of inverse systems

$$0 \longrightarrow (A_i)_i \longrightarrow (A)_i \longrightarrow (A/A_i)_i \longrightarrow 0,$$

since the constant system $(A)_i$ has $\varprojlim_i A = A$ and $\varprojlim_i{}^1 A = 0$. □

The following result summarizes the discussion above.

Corollary A.3. *Let X be a complex of abelian groups, endowed with a Hausdorff filtration $X = X^0 \supseteq X^1 \supseteq X^2 \supseteq \ldots \supseteq X^i \supseteq \ldots$. Then, the complex $\varprojlim_i{}^1 X^i$ is computed by the exact sequence (6) and there is a short exact sequence of abelian groups*

$$0 \longrightarrow \varprojlim_i{}^1 H_{n+1}(X^i) \longrightarrow H_{n+1}\Big(\varprojlim_i{}^1 X^i\Big) \xrightarrow{\partial} \varprojlim_i H_n(X^i) \longrightarrow 0$$

for all n. □

References

[1] Bass, H.: lnjective dimension in Noetherian rings. Trans. Amer. Math. Soc. **102**, 18-29 (1962).
[2] Benson, D.J., Carlson, J.F.: Products in negative cohomology. J. Pure Appl. Algebra **82**, 107-129 (1992)
[3] Cartan, H., Eilenberg, S.: Homological Algebra. Princeton University Press 1956
[4] Celikbas, O., Christensen, L.W., Liang, L., Piepmeyer, G.: Stable homology. arXiv:1409.3605v1
[5] Celikbas, O., Christensen, L.W., Liang, L., Piepmeyer, G.: Complete homology. arXiv:1501.00297v1
[6] Ding, N., Chen, J.: The flat dimensions of injective modules. Manuscripta Math. **78**, 165- 177 (1993)
[7] Emmanouil, I.: Mittag-Leffler condition and the vanishing of $\varprojlim^1$. Topology **35**, 267-271 (1996)
[8] Emmanouil, I.: On the finiteness of Gorenstein homological dimensions. J. Algebra **372**, 376-396 (2012)
[9] Emmanouil, I., Talelli, O.: On the flat length of injective modules. J. London Math. Soc. **84**, 408-432 (2011)
[10] Enochs, E.E., Jenda, O.M.G.: Copure injective resolutions, flat resolvents and dimensions. Comment. Math. Univ. Carolin. **34**, 203-211 (1993)
[11] Gedrich, T.V., Gruenberg, K.W.: Complete cohomological functors on groups. Topology Appl. **25**, 203-223 (1987)
[12] Goichot, F.: Homologie de Tate-Vogel équivariante. J. Pure Appl. Algebra **82**, 39-64 (1992)
[13] Grothendieck, A.: EGA III. Publ. Math. IHES **11** (1961)
[14] Iwanaga, Y.: On rings with finite self-injective dimension II. Tsukuba J. Math. **4**, 107-113 (1980)
[15] Jo, J.H.: Complete homology and related dimensions of groups. J. Group Theory **12**, 431-448 (2009)
[16] Kropholler, P.H.: On groups of type $(\mathrm{FP})_\infty$. J. Pure Appl. Algebra **90**, 55-67 (1993)
[17] Lam, T.Y.: Lectures on Modules and Rings. Grad. Texts Math. **189**, Springer, Berlin 1999
[18] Mislin, G.: Tate cohomology for arbitrary groups via satellites. Topology Appl. **56**, 293-300 (1994)
[19] Nucinkis, B.E.A.: Complete cohomology of arbitrary rings using injectives. J. Pure Appl. Algebra **131**, 297-318 (1998)
[20] Raynaud, M., Gruson, L.: Critères de platitude et de projectivité. Invent. Math. **13**, 1-89 (1971)
[21] Triulzi, M.: Completion constructions in homological algebra and finiteness conditions. Ph.D. Thesis, Swiss Federal Institute of Technology, 1998
[22] Xu, J.: Flat Covers of Modules. Lect. Notes Math. **1634**, Springer, Berlin 1996

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